Reliability Theory Reformulated
Conditional Probability of Failure: A Corollary of the VRPD Theorem

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Reliability Theory Reformulated

Conditional Probability of Failure: A Corollary of the VRPD Theorem

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***Abstract*— The Conditional Probability of Failure (CPF) has been used in reliability theory for reparable units as for non-reparable units in the same way. However, they respond to different stochastic processes. This is why the Reliability Function is replaced by other distribution functions that better fit the events rate curve to obtain a better approximation of the desired estimation.**

The Probability of Failure for non-reparable units is derived here through a proof based on a mental experiment for non-reparable units, avoiding assumptions on the distribution of the random variable, letting the experiment reveals the nature of the phenomenon under test. The proof leads to a Conditional Probability of Failure, presented after a detailed discussion where it is also proved that the hazard rate in the traditional CPF can't be a constant.

***Index Terms*—Conditional Probability of Failure, Failure Rate, Hazard Rate, Reliability Function, Reliability Theory**.

I. INTRODUCTION

TODAY the Reliability Function for reparable and non-reparable units is given by:

$$R(t) = e^{-\lambda t}$$

With: $\lambda = \frac{1}{\theta}$: Constant. And:

$$\theta = \begin{cases} MTBF: & For\ reparable\ units \\ MTTF: & For\ non-reparable\ units \end{cases}$$

In practice, it is necessary to find a probability distribution that fits every case. i.e., a different one from the Reliability Function that was derived for serving that specific purpose, e.g., the Weibull distribution:

$$R(t) = e^{-(\lambda t)^{\beta}}$$

A first reason of the above ill-condition is that the reliability theory starts from an unknown phenomenon, resulting in a biased Reliability Function, specified for the flat section of the device's events rate.

A constant intensity is the exceptional case. The bathtub is a persuasive impression of a regular variable-rate process that requires additional resources beyond the Reliability Function. In other words, the Reliability Theory requires to be reformulated.

This paper proves that:

The probability of a failure event ($P(X=1)$) of a non-reparable device (D) in any interval $t_a \le t < t_b$ along its lifecycle with a known rate function $\lambda(t)$ is given by:

$$\underset{t_a \le T \le t_b}{\overline{P}(X=1)} = \frac{1}{\underset{[t_a,\infty)}{K(t)}} \int_{t_a}^{t_b} \lambda(t)dt$$

With $\overline{P}$: The Conditional Probability, $\lambda(t)$: The events rate function, $K(t)$: The conditional subspace of events of the component D given by:

$$\underset{[t_a,\infty)}{K(t)} = \lim_{t_n \to \infty} \int_{t_a}^{t_n} \lambda(t)dt$$

II. THEOREM PROOF METHOD

We need to build an experiment that represents the

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phenomenon we want to study. The following considerations defines the basis of the subsequent analysis:

1) The failure event in a non-reparable unit occurs just once in the device lifecycle.
2) The random variable must be the failure event, not the time.
3) The random variable, is not Poisson distributed, given that: a) The events rate is variable. b) The desired probability is of a single failure in the analyzed period, it is not of a count of events.
4) The general case of a variable rate event that takes place in time consists of a binomial-distributed random variable.
5) In the case of reparable units (Nino theorem DOI: 10.13140/RG.2.2.24396.28804), the events are independent and non-exclusive. This means that the system may have several failures, where the probability of failure in each time interval is independent of previous events.
6) In the case of non-reparable units, the events are independent and exclusive.

The chapter II contains the analysis on probability of failure of single-failure devices broken down in three sections:

In Section A we demonstrate that the discrete probability of failure is in fact conditional, along with its respective conditional events rate. We also prove that a constant conditional events rate implies a decreasing events rate.

In Section B, we analyze the traditional Reliability Theory from two perspectives, always from the original equations, proving that the Reliability Function only works when the conditional events rate is zero.

In Section C, we derive the continuous Reliability Function from the discrete function previously derived in Section A through two different ways.

In Chapter III we discuss the reasons that made famous the traditional Reliability Theory despite its short scope and application constraints, and the well-known memoryless attribute.

The Appendix contains a short explanation on how to determine the early-failure period in the events rate function.

The proof is based on a mental experiment of m single-failure devices starting their lifecycle at the same time.

A. The conditional events rate may be constant if and only if the events rate is a decreasing function of time

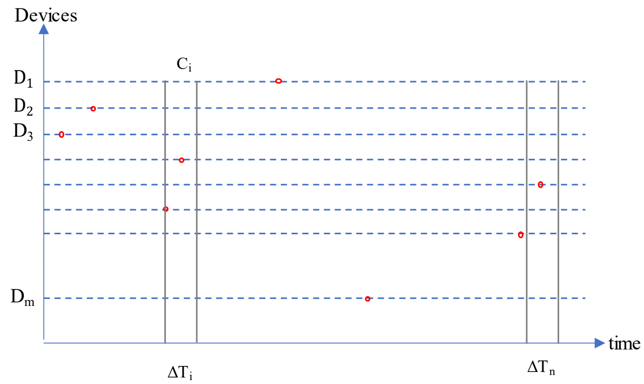

Fig. 1. Failures in time of m cloned devices.

Suppose an experiment with m clones of the unit D, starting their lifecycle at the same time ($t = 0$).

Let us suppose that we divide the lifecycle in n intervals of the same size.

Let us take a timeframe at T_i out of the component lifecycle

$$\Delta T_i = t_b - t_a \tag{1}$$

With:

$$|\Delta T_i| = \Delta T \ \forall i \tag{2}$$

$$c_i = \sum_{j=1}^{m} c_{ij} \tag{3}$$

With c_i: the sum of all the events of the m units in ΔT_i

Knowing that every device fails just once in its lifecycle, there will be a count of m failures at the end of the lifecycle. Therefore:

$$m = \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij} \tag{4}$$

Also:

$$c_{ij} = \begin{cases} 0, & \textit{when there is no event} \\ 1, & \textit{when there is an event} \end{cases} \tag{5}$$

The clones will fail gradually in time. Thus, in ΔT_1 it will be a count of C_1 failures (components out), and despite the size of ΔT it happens that:

$$0 \leq \frac{C_1}{m} \leq 1 \tag{6}$$

Equation (6) is the *empirical probability of interruption* of the device D, as it is stablished by the theorem of large numbers for Bernoulli trials:

$$\lim_{m\to\infty} P\left(\left|\frac{c_1}{m} - p_1\right| \geq \epsilon\right) = 0 \tag{7}$$

With p_1: *Probability of a Bernoulli event* in Δt_i.

An equivalent identity is given by the general low of large numbers which states that:

$$\lim_{m\to\infty} \frac{c_1}{m} = p_1 \tag{8}$$

We can say that the interruption rate of D in ΔT_1 is given by:

$$\lim_{m\to\infty} \left(\frac{c_1}{m\Delta T}\right) = \frac{p_1}{\Delta \mathrm{T}} = \lambda_1 \tag{9}$$

Or what is the same:

$$p_1 = \Delta \mathrm{T}\lambda_1 \tag{10}$$

Therefore, for any $\Delta \mathrm{T}_i$ it happens that:

$$p_1 = \frac{c_1}{m}, \qquad p_2 = \frac{c_2}{m - c_1}, \qquad p_3 = \frac{c_3}{m - c_1 - c_2},$$

$$\ldots p_i = \frac{c_i}{m - \sum_{k=1}^{i-1} c_k} \tag{11}$$

With:

$$c_i = \sum_{j=1}^{m} c_{ij} \quad \forall i, i = 1,2,3,\ldots,n$$

Equation (11) corresponds to a conditional probability. Thus, for any interval $\Delta \mathrm{T}_i = t_b - t_a$, from eq11 we have:

$$\overline{p_i} = \frac{c_i}{m - \sum_{k=1}^{i-1} c_k} \tag{12}$$

Or what is the same:

$$\overline{p_i} = \frac{c_i}{\sum_{k=i}^{n} c_k} \tag{13}$$

With $\overline{p_i}$: The discrete *conditional probability of failure* in $\Delta \mathrm{T}_i$, while the denominator of (13) is its *conditional subspace of events*.

From (10) and (13) we know that:

$$\frac{\overline{p_i}}{\Delta \mathrm{T}} = \overline{\lambda_i}\ \forall k \tag{14}$$

Or:

$$\overline{\lambda_i} = \frac{c_i}{\Delta \mathrm{T}(\sum_{k=i}^{n} c_k)} \tag{15}$$

With $\bar{\lambda}_i$: Conditional Events Rate or Conditional Intensity. Now, a constant $\bar{\lambda}$ should satisfy the following condition:

$$\frac{c_{i+1}}{\sum_{k=i+1}^{n} c_k} = \frac{c_i}{\sum_{k=i}^{n} c_k} \tag{16}$$

This is:

$$\frac{c_{i+1}}{c_i} = 1 - \frac{c_i}{\sum_{k=i}^{n} c_k} \tag{17}$$

Thus:

$$\overline{p_i} = 1 - \frac{c_{i+1}}{c_i} \tag{18}$$

Therefore, it must happen that:

$$\frac{c_i}{m\Delta \mathrm{T}} \geq \frac{c_{i+1}}{m\Delta \mathrm{T}}\ \forall i$$

This is:

$$\lambda_i \geq \lambda_{i+1} \tag{19}$$

But, if $\lambda_i = \lambda_{i+1}$. This is, if $c_i = c_{i+1}$, in eq17 it yields that: $\bar{\lambda}_i = 0\ \forall i$.

Then (19) becomes:

$$c_i > c_{i+1}\ \forall i \quad (20)$$

Preliminary Conclusion:

A non-reparable unit may have a conditional constant rate $(\bar{\lambda})$ *if and only if the events rate* $(\lambda(t))$ *is a decreasing function of time*.

B. The well-known conditional probability of failure and two pathways

The Reliability Theory [5] is based on the following postulates:

$$F(t) + R(t) = 1 = R(0) \quad (21)$$

$R(t)$ is defined as the Probability of Failure in $\mathrm{T} > t$, knowing that the unit has not failed until $\mathrm{T} = t$.

Equation (21) strictly says that the device may fail just once. So far, the theory has not been defined a subspace of events different from $[0, \infty)$. This means that a single-failure event is an implicit postulate in (21).

The theory derives the Conditional Probability of Failure $\bar{p}$ in a time interval $\Delta\mathrm{T}$ as follows:

$$\bar{p}_{\Delta\mathrm{T}} = \frac{F(t+\Delta\mathrm{T}) - F(t)}{R(t)} \quad (22)$$

Thus, the *conditional intensity* is nothing but:

$$\bar{\lambda}_{\Delta\mathrm{T}} = \frac{F(t+\Delta\mathrm{T}) - F(t)}{R(t)\Delta\mathrm{T}} \quad (23)$$

With $\bar{\lambda}$: constant for every $\Delta\mathrm{T}$.

If we take the limit when $\Delta\mathrm{T} \to 0$, we have:

$$\frac{dF(t)}{R(t)} = \bar{\lambda} dt \quad (24)$$

There are two pathways from here:

1) *The differential of conditional probability of failure*

From (24) we may write:

$$d\bar{F}(t) = \bar{\lambda} dt \quad (25)$$

Now, the *Cumulative Conditional Probability of Failure* will result from integrating both sides of (25):

$$\bar{F}(t) = \bar{\lambda} t \quad (26)$$

With $\bar{F}(t)$: The *Cumulative Conditional Probability of Failure* in $\mathrm{T} = t$.

2) *The exponential Reliability Function approach*

Equation (24) me be written as:

$$-\frac{dR(t)}{R(t)} = \bar{\lambda} dt \quad (27)$$

Integrating both sides of (27) we have:

$$R(t) = e^{-\bar{\lambda} t} \quad (28)$$

With $R(t)$: The reliability function. Which from (21):

$$F(t) = 1 - e^{-\bar{\lambda} t} \quad (29)$$

Equation (29) is the *Cumulative Probability of Failure*.

Now, the Cumulative Conditional Probability of Failure is given by:

$$\frac{F(t)}{R(0)} = F(t)$$

This means that, the left side of (29), is nothing but the Cumulative Conditional Probability of Failure, and may be written:

$$\bar{F}(t) = 1 - e^{-\bar{\lambda} t} \quad (30)$$

Now, we shall equate (26) and (30):

$$\bar{\lambda} t = 1 - e^{-\bar{\lambda} t} \quad (31)$$

Now, if we apply the derivatives in both sides of (31) we arrive to the same result as:

$$\left(1 - \bar{\lambda} t\right)^{-\frac{1}{\bar{\lambda} t}} = e \ \ \forall t \quad (32)$$

Now, making $-\overline{\lambda}t = \frac{1}{x}$ we have that:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \tag{33}$$

It results in $\overline{\lambda} = 0$.

Conclusion:
In Reliability Theory, for any interval $\Delta\mathrm{T} = [0,t)$, (30) holds only if the conditional events rate $\overline{\lambda} = 0$.

C. The proof of the Conditional Probability of Failure for non-reparable devices

We will derive the Conditional Probability of Failure using two different approaches. The first approach will be through the discrete analysis we started in *section A*. The second approach will be applying the concept of conditional probability as a particular case of VRPD Theorem [1].

1) *Through the Discrete analysis*

Recalling (12):

$$\overline{p_i} = \frac{c_i}{m - \sum_{k=1}^{i-1} c_k}$$

Normalizing the count of events, dividing the numerator and denominator by m, we have:

$$\overline{p_i} = \frac{\frac{c_i}{m}}{\left(1 - \sum_{k=1}^{i-1}\frac{c_k}{m}\right)} \tag{34}$$

We know that:

$$\frac{c_i}{m} = p_i = \lambda_i\Delta\mathrm{T} \tag{35}$$

Therefore (34) becomes:

$$\overline{p_i} = \frac{\lambda_i\Delta\mathrm{T}}{\left(1 - \sum_{k=1}^{i-1}\lambda_k\Delta\mathrm{T}\right)}$$

Or what is the same:

$$\overline{p_i} = \frac{\lambda_i\Delta\mathrm{T}}{\left(\sum_{k=i}^{\infty}\lambda_k\Delta\mathrm{T}\right)} \tag{36}$$

We can always find a continuous function by parts such that:

$$\lambda_i = \zeta_i \tag{37}$$

If we take the events rate λ_i (events per unit of time) instead of the normalized events rate, $\frac{c_i}{m\Delta\mathrm{T}}$, we may always find a function $\zeta_i(t)$ that, according to the Mean Value Theorem for integrals results in:

$$\lambda_i\Delta\mathrm{T} = \frac{1}{r}\int_{t_{i-1}}^{t_i}\zeta_i(t)dt \tag{38}$$

With r: The proportionality factor for fitting any time unit of the intensity function.

Now. If instead of a continuous-by-parts function we have a function $\lambda(t)$, continuous in $[0,\infty)$, such that:

$$\lambda(t) = \sum_{i=1}^{\infty}\zeta_i(t) \tag{39}$$

Then:

$$\int_{t_{i-1}}^{t_i}\zeta_i(t)dt = \int_{t_{i-1}}^{t_i}\lambda(t)dt \tag{40}$$

As $\Delta\mathrm{T}$ can be as large as needed (according to (6)), let us make $t_{i-1} = t_a$, $t_i = t_b$.

Then, we may write:

$$\lambda_i\Delta\mathrm{T} = \frac{1}{r}\int_{t_a}^{t_b}\lambda(t)dt \tag{41}$$

With (41) in (36), we have:

$$\overline{P}_{t_a\le T\le t_b}(X=1) = \frac{1}{\int_{t_a}^{\infty}\lambda(t)dt}\int_{t_a}^{t_b}\lambda(t)dt \tag{42}$$

2) *Applying Conditional Probability to the Nino Theorem*

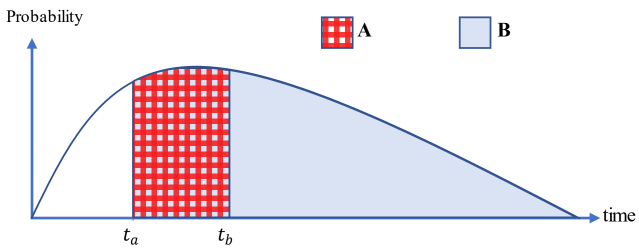

Fig. 2. Regions of a probability for a generic Subspace B.

We may apply the VRPD theorem [1] to single-failure events if we ensure that the variable is Bernoulli distributed with replacement. This means that the probability must always be calculated in intervals $\Delta T_i[t_a, t_b)$, with $t_a = 0$. Because when $t_a \neq 0$, we will have $m - \sum_{k=1}^{i-1} c_k$ operating devices, which results in a conditional subspace of events, that takes us to the Conditional Probability of Failure obtained in the first approach. In other words, VRPD theorem can only be applied to a single-event process (a Bernoulli distributed variable) in terms of a *cumulative probability*. i.e., a particular case of VRPD.

Let us apply the VRPD theorem:

$$\underset{0 \leq T < t}{P}(X = 1) = \frac{1}{W} \int_0^t \lambda(x) dx$$

With:

$$W = \int_0^\infty \lambda(t) dt$$

Now:

$$P(A \cap B) = P(A) = \frac{1}{W} \left[\int_0^{t_b} \lambda(t) dt - \int_0^{t_a} \lambda(t) dt \right] \qquad (43)$$

$$P(B) = \frac{1}{W} \left[\int_0^{\infty} \lambda(t) dt - \int_0^{t_a} \lambda(t) dt \right] \qquad (44)$$

According to the conditional probability theory we know:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad (45)$$

Now, plugging (43) and (44) into (45), we have:

$$\underset{t_a \leq T < t_b}{\overline{P}}(X = 1) = \frac{1}{\int_{t_a}^{\infty} \lambda(t) dt} \int_{t_a}^{t_b} \lambda(t) dt \qquad (46)$$

Q.E.D.

III. The Convenience of the Exponential Function in Reliability Theory

In chapter II we proved that the intensity λ must be a decreasing function, while the conditional intensity $\overline{\lambda}$ cannot be a constant.

Here after, Conditional Probability of Failure corresponds to (46).

In the following reasoning we will take a particular intensity function, which fulfills the restriction posed by (19), i.e., a deceasing function.

A. The Conditional Probability of Failure when $\lambda(t) = Ae^{-\mu t}$

From (47), if we take the Cumulative Conditional Probability of Failure with the exponential function, we have:

$$\underset{0 \leq T \leq t}{\overline{P}}(X = 1) = 1 - e^{-\mu t} \qquad (47)$$

Equation (47) is quite like (30), but here, μ is just a parameter that must be defined according to every experiment.

If we want to calculate the Conditional Probability of Failure in a specific time interval $[t_a, t_b)$, we have from (46):

$$\underset{t_a \leq T \leq t_b}{\overline{P}}(X = 1) = 1 - e^{-\mu(t_b - t_a)} \qquad (48)$$

B. The memoryless attribute when $\lambda(t) = Ae^{-\mu t}$

Let us prove that if λ is the exponential function, then $\overline{\mathrm{P}}$ is constant:

This is, taking $\overline{P}$: constant, we have:

$$1 - e^{-\mu \Delta t_{i-1}} = 1 - e^{-\mu \Delta t_i} \iff$$

$$1 - e^{-\mu(t_b - t_a)} = 1 - e^{-\mu(t_c - t_b)} \iff$$

$$t_b = \frac{t_a + t_c}{2}$$

IV. Conclusions

The practical need of replacing the reliability function by other distributions like the Weibull distribution or Poisson variations is due to the independence of the reliability theory from the phenomenon it must explain. The reasons were provided along this paper and were condensed in the conclusions of every section.

Now, when the events rate is the exponential function:

1) The *Conditional Probability of Failure remains constant for equal time intervals:* This is the memoryless attribute
2) This confirms that it is the exponential events rate that must fulfill both the restriction of the eq19 and the memoryless condition.
3) The exponential function derived from the reliability theory with its memoryless condition is a convenient coincidence but unworkable when, using $\overline{\lambda}$ as if it was λ, as it was demonstrated along chapter II.

Appendix

A helpful procedure to determine the early-failure limits when the intensity function is the exponential function.

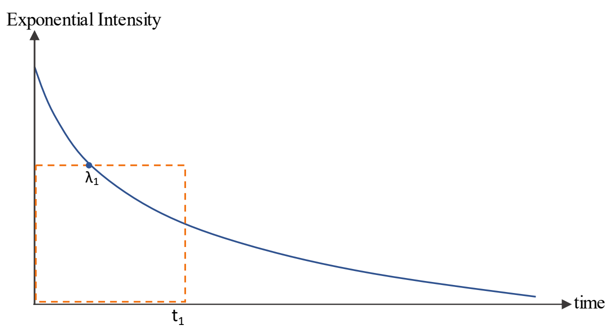

Fig. 3. Defining the limits for early-failure period.

From the Mean Value Theorem for integrals, we may always find a value λ_1, such that:

$$\int_0^{t_1} Ae^{-\mu t}\,dt = t_1\lambda_1$$

From (8) and (10), we know that:

$$t_1\lambda_1 = \frac{c_1}{m} = \overline{P}_1 \qquad (49)$$

Now, considering the case when:

$$\int_0^{\infty} Ae^{-\mu t}\,dt = 1$$

$$\Rightarrow \lambda(t) = \mu e^{-\mu t} \qquad (50)$$

$$\Rightarrow \underset{0\le t\le t_1}{\overline{P}}(X=1) = 1 - e^{-\mu t_1} \qquad (51)$$

According to (49), if we have the normalized count of events of *m* units in the early-failure period (up to t_1), we simply put this value in the left side of (51) to determine μ.

And, according to (49), if we divide the left side of (51) by t_1, we get the average events rate corresponding to this period.

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He created a consultancy firm of IT, Risk Management and Business Continuity in 2012 with the purpose of providing new ideas and academic support to the concepts underlying the consultancy practices.

Mr. Aristizábal is a CBCP professional; DRII associate since 2009. In 2018, when seeking a solution to calculate the probability of interruption of organizational processes, he discovered the probability distribution function for any variable-rate process (VRPD-Nino theorem) approved by the academic community before the International Symposium of Statistics, U. Nacional de Colombia – 2019. along with another theorem and mathematical models applied to Business Continuity. With more than fifteen years of experience in telecommunications networks and Business Continuity in more than seven industries, Alejandro Aristizabal is the author of "Probability of Interruption and Business Impact", a book that approaches the Business Continuity from a mathematical perspective.